Chapter 3

1. Describe an algorithm that takes a list of n integers $a_1,a_2,\ldots,a_n$ and finds the number of integers each greater than five in the list.

   Ans: procedure *greaterthanfive*($a_1,\ldots,a_n$: integers)
   $answer := 0$
   for $i := 1$ **to** n
   if $a_i > 5$ **then** $answer := answer + 1$.

2. Describe an algorithm that takes a list of integers $a_1,a_2,\ldots,a_n$ ($n \geq 2$) and finds the second-largest integer in the sequence.

   Ans: procedure *secondlargest*($a_1,\ldots,a_n$: integers)
   $largest := a_1$
   $secondlargest := a_2$
   if $a_2 > a_1$ **then**
   begin
   $secondlargest := a_1$
   $largest := a_2$
   end
   if $n = 2$ **then**
   stop
   for $i := 3$ **to** n
   if $a_i > largest$ **then**
   begin
   $secondlargest := largest$
   $largest := a_i$
   end
   if ($a_i > secondlargest$ **and** $a_i \leq largest$) **then**
   $secondlargest := a_i$.

3. Describe an algorithm that takes a list of n integers ($n \geq 1$) and finds the location of the last even integer in the list, or returns 0 if there are no even integers in the list.

   Ans: procedure *lasteven*($a_1,\ldots,a_n$: integers)
   $location := 0$
   for $i := 1$ **to** n
   if $2 \mid a_i$ **then** $location := i$.

4. Describe an algorithm that takes a list of n integers ($n \geq 1$) and finds the average of the largest and smallest integers in the list.

   Ans: procedure *avgmaxmin*($a_1,\ldots,a_n$: integers)

   $max := a_1$
   $min := a_1$
   for $i := 2$ **to** n
   begin
    if $a_i > max$ **then** $max := a_i$
    if $a_i < min$ **then** $min := a_i$
   end
   $avg := (max + min)/2$.

5. Describe in words how the binary search works.

   Ans: To search for x in an ordered list $a_1,\ldots,a_n$, find the "midpoint" of the list and choose the appropriate half of the list. Continue until the list consists of one element. Either this element is x, or else x is not in the list.

6. Show how the binary search algorithm searches for 27 in the following list: 5 6 8 12 15 21 25 31.

   Ans: The consecutive choices of sublists of the original list are: 15 21 25 31, 25 31, and 25. Since $27 \neq 25$, the integer 25 is not in the list.

7. You have supplies of boards that are one foot, five feet, seven feet, and twelve feet long. You need to lay pieces end-to-end to make a molding 15 feet long and wish to do this using the fewest number of pieces possible. Explain why the greedy algorithm of taking boards of the longest length at each stage (so long as the total length of the boards selected does not exceed 15 feet) does not give the fewest number of boards possible.

   Ans: The greedy algorithm first chooses a 12-foot-long board, and then three one-foot-long boards. This requires four boards. But only three boards are needed: each five feet long.

8. Use the definition of big-oh to prove that $1^2 + 2^2 + \ldots + n^2$ is $O(n^3)$.

   Ans: $1^2 + 2^2 + \ldots + n^2 \leq n^2 + n^2 + \ldots + n^2 = n \cdot n^2 = n^3$.

9. Use the definition of big-oh to prove that $\dfrac{3n-8-4n^3}{2n-1}$ is $O(n^2)$.

   Ans: $\dfrac{3n-8-4n^3}{2n-1} \leq \dfrac{3n^3+8n^3+4n^3}{2n-n} = \dfrac{15n^3}{n} = 15n^2$ if $n \geq 1$.

10. Use the definition of big-oh to prove that $1^3 + 2^3 + \ldots + n^3$ is $O(n^4)$.

    Ans: $1^3 + 2^3 + \ldots + n^3 \leq n^3 + n^3 + \ldots + n^3 = n \cdot n^3 = n^4$.

11. Use the definition of big-oh to prove that $\dfrac{6n+4n^5-4}{7n^2-3}$ is $O(n^3)$.

Ans: $\dfrac{6n+4n^5-4}{7n^2-3} \le \dfrac{6n^5+4n^5}{7n^2-n^2} = \dfrac{10n^5}{6n^2} = \dfrac{5}{3}n^3$, **if** $n \ge 2$.

12. Use the definition of big-oh to prove that $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + (n-1) \cdot n$ is $O(n^3)$.

Ans: $1 \cdot 2 + 2 \cdot 3 + \ldots + (n-1) \cdot n \le (n-1) \cdot n + (n-1) \cdot n + \ldots + (n-1) \cdot n = (n-1)^2 \cdot n \le n^3$.

13. Let $f(n) = 3n^2 + 8n + 7$. Show that $f(n)$ is $O(n^2)$. Find C and k from the definition.

Ans: $f(n) \le 3n^2 + 8n^2 + 7n^2 = 18n^2$ **if** $n \ge 1$; therefore $C = 18$ and $k = 1$ can be used.

Use the following to answer questions 14-19:

In the questions below find the best big-oh function for the function. Choose your answer from among the following:

$$1,\quad \log_2 n,\quad n,\quad n\log_2 n,\quad n^2,\quad n^3, \ldots,\quad 2^n,\quad n!.$$

14. $f(n) = 1 + 4 + 7 + \ldots + (3n+1)$.

Ans: n^2.

15. $g(n) = 1 + 3 + 5 + 7 + \ldots + (2n-1)$.

Ans: n^2.

16. $\dfrac{3-2n^4-4n}{2n^3-3n}$.

Ans: n.

17. $f(n) = 1 + 2 + 3 + \ldots + (n^2-1) + n^2$.

Ans: n^4.

18. $\lceil n+2 \rceil \cdot \lceil n/3 \rceil$.

Ans: n^2.

19. $3n^4 + \log_2 n^8$.

Ans: n^4.

20. Show that $\sum_{j=1}^{n} (j^3 + j)$ is $O(n^4)$.

Ans: $\sum_{j=1}^{n} (j^3 + j) \le \sum_{j=1}^{n} (n^3 + n^3) = n \cdot 2n^3 = 2n^4$.

21. Show that $f(x) = (x + 2)\log_2(x^2 + 1) + \log_2(x^3 + 1)$ is $O(x\log_2 x)$.
Ans: $\log_2(x^2 + 1)$ and $\log_2(x^3 + 1)$ are each $O(\log_2 x)$. Thus each term is $O(x\log_2 x)$, and hence so is the sum.

22. Find the best big-O function for $n^3 + \sin n^7$.
Ans: n^3.

23. Find the best big-O function for $\dfrac{x^3+7x}{3x+1}$.
Ans: x^2.

24. Prove that $5x^4 + 2x^3 - 1$ is $\Theta(x^4)$.
Ans: $5x^4 + 2x^3 - 1$ is $O(x^4)$ since $|5x^4 + 2x^3 - 1| \le |5x^4 + 2x^4| \le 7|x^4|$ (if $x \ge 1$). Also, x^4 is $O(5x^4 + 2x^3 - 1)$ since $|x^4| \le |5x^4 + x^3| \le |5x^4 + 2x^3 - 1|$ (if $x \ge 1$).

25. Prove that $\dfrac{x^3+7x^2+3}{2x+1}$ is $\Theta(x^2)$.
Ans: $\dfrac{x^3+7x^2+3}{2x+1}$ is $O(x^2)$ since $\dfrac{x^3+7x^2+3}{2x+1} \le \dfrac{x^3+7x^3+3x^3}{2x} = \dfrac{11x^3}{2x} = \dfrac{11}{2}x^2$ (if $x \ge 1$).
Also, x^2 is $O\left(\dfrac{x^3+7x^2+3}{2x+1}\right)$ since $x^2 = \dfrac{x^3}{x} \le \dfrac{x^3+7x}{2x} \le \dfrac{x^3+7x+3}{2x+1} \le \dfrac{x^3+7x^2+3}{2x+1}$ (if $x \ge 1$).

26. Prove that $x^3 + 7x + 2$ is $\Omega(x^3)$.
Ans: $x^3 + 7x + 2 \ge 1 \cdot x^3$ (if $x \ge 1$).

Use the following to answer questions 27-37:
In the questions below find the "best" big-oh notation to describe the complexity of the algorithm. Choose your answers from the following:

$$1,\ \log_2 n,\ n,\ n\log_2 n,\ n^2,\ n^3, \ldots,\ 2^n,\ n!.$$

27. A binary search of n elements.
Ans: $\log_2 n$.

28. A linear search to find the smallest number in a list of n numbers.
Ans: n.

29. An algorithm that lists all ways to put the numbers $1,2,3,\ldots,n$ in a row.
Ans: $n!$.

30. An algorithm that prints all bit strings of length n.
Ans: 2^n.