Preparation for the Final Exam

1. Find a proposition with three variables p, q, and r that is true when p and r are true and q is false, and false otherwise
   Ans: p and not q and r.

2. Find a proposition using only p,q,$\neg$ and the connective $\vee$ with the given truth table.

p	q	?
T	T	F
T	F	T
F	T	T
F	F	F

   Ans: $\neg(\neg p \vee q) \vee \neg(p \vee \neg q)$.

3. Determine whether this proposition is a tautology: $((p \to \neg q) \wedge q) \to \neg p$.
   Ans: Yes.

Use the following to answer questions 4-6:

In the questions below write the statement in the form "If ..., then ...."

4. x is even only if y is odd.
   Ans: If x is even, then y is odd.

5. A implies B.
   Ans: If A, then B.

6. It is hot whenever it is sunny.
   Ans: If it is sunny, then it is hot.

7. Find three subsets of {1,2,3,4,5,6,7,8,9} such that the intersection of any two has size 2 and the intersection of all three has size 1.
   Ans: For example, {1,2,3}, {2,3,4}, {1,3,4}.

8. Suppose U = {1, 2, ..., 9}, A = all multiples of 2, B = all multiples of 3, and C = {3, 4, 5, 6, 7}.
Find C - (B - A).
   Ans: {4,5,6,7}.

9. Describe an algorithm that takes a list of n integers ($n \geq 1$) and finds the average of the largest and smallest integers in the list.

Ans: procedure *avgmaxmin*($a_1, \ldots, a_n$: integers)
max: = a_1
min: = a_1
for i : = 2 **to** n
begin
 if $a_i > max$ **then** $max := a_i$
 if $a_i < min$ **then** $min := a_i$
end
avg: =(*max* + *min*)/2.

10. Use the definition of big-oh to prove that $1^2 + 2^2 + \ldots + n^2$ is $O(n^3)$.

Ans: $1^2 + 2^2 + \ldots + n^2 \leq n^2 + n^2 + \ldots + n^2 = n \cdot n^2 = n^3$.

11.Use the definition of big-oh to prove that $\dfrac{3n-8-4n^3}{2n-1}$ is $O(n^2)$.

Ans: $\dfrac{3n-8-4n^3}{2n-1} \leq \dfrac{3n^3+8n^3+4n^3}{2n-n} = \dfrac{15n^3}{n} = 15n^2$ if $n \geq 1$.

12. Use the Principle of Mathematical Induction to prove that

$$1-2+2^2-2^3+\ldots+(-1)^n 2^n = \frac{2^{n+1}(-1)^n+1}{3}$$ for all positive integers n.

Ans: $P(1)$: $1-2=\dfrac{2^2(-1)+1}{3}$, which is true since both sides are equal to -1. $P(k) \rightarrow P(k+1)$:

$$1-2+2^2+\ldots+(-1)^{k+1}2^{k+1} = \frac{2^{k+1}(-1)^k+1}{3}+(-1)^{k+1}2^{k+1} = \frac{2^{k+1}(-1)^k+1+3(-1)^{k+1}2^{k+1}}{3}$$

$$= \frac{2^{k+1}(-1)^k(1+3(-1))+1}{3} = \frac{2^{k+1}(-1)^k(-2)+1}{3} = \frac{2^{k+2}(-1)^{k+1}+1}{3}.$$

13. Use the Principle of Mathematical Induction to prove that $1 + 2^n \leq 3^n$ for all $n \geq 1$.

Ans: $P(1)$: $1 + 2^1 \leq 3^1$, which is true since both sides are equal to 3. $P(k) \rightarrow P(k+1)$: $1 + 2^{k+1} = (1 + 2^k) + 2^k \leq 3^k + 2^k \leq 3^k + 3^k = 2 \cdot 3^k < 3 \cdot 3^k = 3^{k+1}$.

14. Three coins are tossed.

(a) List the elements in the sample space.

(b) Find the probability that exactly two heads show.

Ans: (a) HHH, HHT, HTH, THH, HTT, THT, TTH, TTT. (b) 3/8.

15. Suppose you and a friend each choose at random an integer between 1 and 8. For example, some possibilities are (3,7), (7,3), (4,4), (8,1), where your number is written first and your friend's number second. Find

(a) p(you pick 5 and your friend picks 8).

(b) p(sum of the two numbers picked is < 4).

(c) p(both numbers match).

(d) p(the sum of the two numbers is a prime).

(e) p(your number is greater than your friend's number).

Ans: (a) 1/64. (b) 3/64. (c) 8/64. (d) 23/64. (e) 28/64.

In the questions below suppose you have a class with 30 students — 10 freshmen, 12 sophomores, and 8 juniors.

16. You pick one student at random. What is the probability that the student is not a junior?

Ans: 22/30.

17. You pick two students at random, one at a time. What is the probability that both are freshmen?

Ans: $(10 \cdot 9)/(30 \cdot 29)$.

18. You pick two students at random, one at a time. What is the probability that the second student is a freshman, given that the first is a freshman?

Ans: 9/29.

19. In a certain lottery game, three distinct numbers between 10 and 25 (inclusive) are chosen as the winning numbers. What is the probability that the winning numbers are all composite numbers.

Ans: $\binom{11}{3} / \binom{16}{3}$.