

1. Prove or disprove: $A - (B \cap C) = (A - B) \cup (A - C)$.

Ans: True, since

$$A - (B \cap C) = A \cap \overline{B \cap C} = A \cap (\overline{B} \cup \overline{C}) = (A \cap \overline{B}) \cup (A \cap \overline{C}) = (A - B) \cup (A - C).$$

2. Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$ by giving a containment proof (that is, prove that the left side is a subset of the right side and that the right side is a subset of the left side).

Ans: $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$: Let $x \in \overline{A \cap B}$. $\therefore x \notin A \cap B$, $\therefore x \notin A$ or $x \notin B$, $\therefore x \in \overline{A}$ or $x \in \overline{B}$, $\therefore x \in \overline{A} \cup \overline{B}$. Reversing the steps shows that $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$.

3. Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$ by giving an element table proof.

Ans: The columns for $\overline{A \cap B}$ and $\overline{A} \cup \overline{B}$ match: each entry is 0 if and only if A and B have the value 1.

4. Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$ by giving a proof using logical equivalence.

$$\overline{A \cap B} = \{x | x \in \overline{A \cap B}\}$$

$$= \{x | x \notin A \cap B\}$$

$$= \{x | \neg(x \in A \cap B)\}$$

$$= \{x | \neg(x \in A \wedge x \in B)\}$$

Ans:

$$= \{x | \neg(x \in A) \vee \neg(x \in B)\}$$

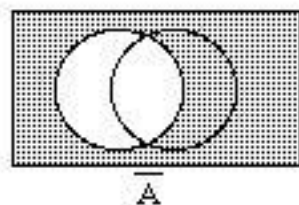
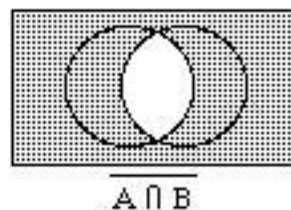
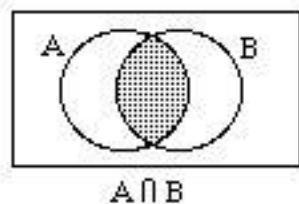
$$= \{x | x \notin A \vee x \notin B\}$$

$$= \{x | x \in \overline{A} \vee x \in \overline{B}\}$$

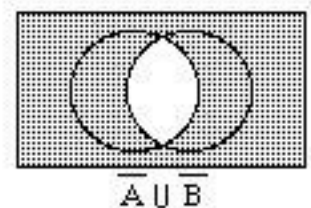
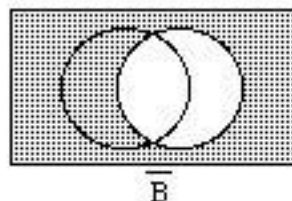
$$= \{x | x \in \overline{A} \cup \overline{B}\} = \overline{A} \cup \overline{B}$$

5. Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$ by giving a Venn diagram proof.

Ans:



∪



6. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ by giving a containment proof (that is, prove that the left side is a subset of the right side and that the right side is a subset of the left side).

Ans: $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$: Let $x \in A \cap (B \cup C)$. $\therefore x \in A$ and $x \in B \cup C$, $\therefore x \in A$ and $x \in B$, or $x \in A$ and $x \in C$, $\therefore x \in (A \cap B) \cup (A \cap C)$. Reversing the steps gives the opposite containment.

7. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ by giving an element table proof.

Ans: Each set has the same values in the element table: the value is 1 if and only if A has the value 1 and either B or C has the value 1.

8. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ by giving a proof using logical equivalence.

$$A \cap (B \cup C) = \{x | x \in A \cap (B \cup C)\}$$

$$= \{x | x \in A \wedge x \in (B \cup C)\}$$

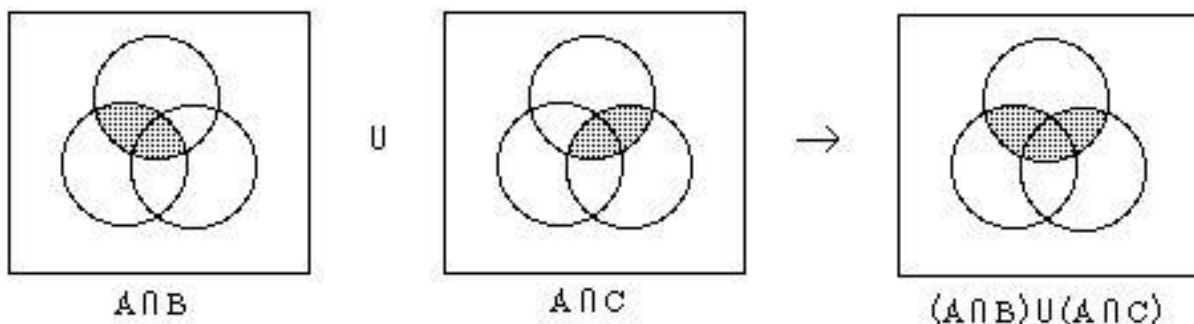
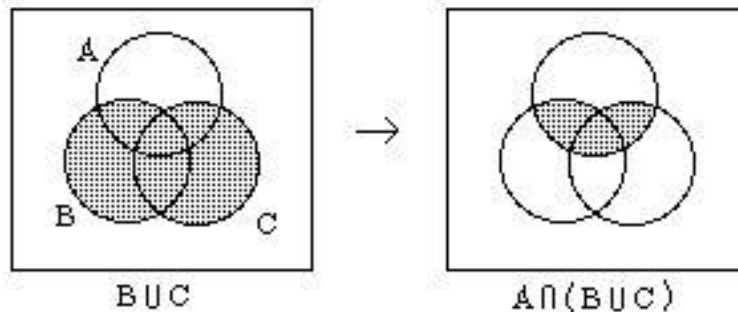
$$= \{x | x \in A \wedge (x \in B \vee x \in C)\}$$

Ans: $= \{x | (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)\}$

$$= \{x | x \in A \cap B \vee x \in A \cap C\}$$

$$= \{x | x \in (A \cap B) \cup (A \cap C)\}$$

$$= (A \cap B) \cup (A \cap C)$$



9. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ by giving a Venn diagram proof.

Ans:

In the questions below determine whether the set is finite or infinite. If the set is finite, find its size.

73. $\{x \mid x \in \mathbf{Z} \text{ and } x^2 < 10\}$.

Ans: 7.

74. $P(\{a,b,c,d\})$, where P denotes the power set.

Ans: 16.

75. $\{1,3,5,7,\dots\}$.

Ans: Infinite.

76. $A \times B$, where $A = \{1,2,3,4,5\}$ and $B = \{1,2,3\}$.

Ans: 15.

77. $\{x \mid x \in \mathbf{N} \text{ and } 9x^2 - 1 = 0\}$.

Ans: 0.

78. $P(A)$, where A is the power set of $\{a,b,c\}$.

Ans: 256.

79. $A \times B$, where $A = \{a,b,c\}$ and $B = \emptyset$.

Ans: 0.

80. $\{x \mid x \in \mathbf{N} \text{ and } 4x^2 - 8 = 0\}$.

Ans: 0.

81. $\{x \mid x \in \mathbf{Z} \text{ and } x^2 = 2\}$.

Ans: 0.

82. $P(A)$, where $A = P(\{1,2\})$.

Ans: 16.

83. $\{1,10,100,1000,\dots\}$.

Ans: Infinite.

84. $S \times T$, where $S = \{a,b,c\}$ and $T = \{1,2,3,4,5\}$.

Ans: 15.

85. $\{x \mid x \in \mathbf{Z} \text{ and } x^2 < 8\}$.

Ans: 5.

86. Prove that between every two rational numbers a/b and c/d

(a) there is a rational number.

(b) there are an infinite number of rational numbers.

Ans: (a) Assume $\frac{a}{b} \leq \frac{c}{d}$. Then $\frac{a}{b} \leq \frac{\frac{a}{b} + \frac{c}{d}}{2} = \frac{ad + bc}{2bd} \leq \frac{c}{d}$.

(b) Assume $\frac{a}{b} < \frac{c}{d}$. Let m_1 be the midpoint of $\left[\frac{a}{b}, \frac{c}{d}\right]$. For $i > 1$ let m_i be the midpoint of $\left[\frac{a}{b}, m_{i-1}\right]$.

87. Prove that there is no smallest positive rational number.

Ans: If $0 < \frac{a}{b}$, then $0 < \dots < \frac{a}{4b} < \frac{a}{3b} < \frac{a}{2b} < \frac{a}{b}$.

Use the following to answer questions 88-96:

In the questions below determine whether the rule describes a function with the given domain and codomain.

88. $f: \mathbf{N} \rightarrow \mathbf{N}$ where $f(n) = \sqrt{n}$.

Ans: Not a function; $f(2)$ is not an integer.

89. $h: \mathbf{R} \rightarrow \mathbf{R}$ where $h(x) = \sqrt{x}$.

Ans: Function.

90. $g: \mathbf{N} \rightarrow \mathbf{N}$ where $g(n) = \text{any integer} > n$.

Ans: Not a function; $g(1)$ has more than one value.

91. $F: \mathbf{R} \rightarrow \mathbf{R}$ where $F(x) = \frac{1}{x-5}$.

Ans: Not a function; $F(5)$ not defined.

92. $F: \mathbf{Z} \rightarrow \mathbf{R}$ where $F(x) = \frac{1}{x^2-5}$.

Ans: Function.

93. $F: \mathbf{Z} \rightarrow \mathbf{Z}$ where $F(x) = \frac{1}{x^2-5}$.

Ans: Not a function; $F(1)$ not an integer.

94. $G : \mathbf{R} \rightarrow \mathbf{R}$ where $G(x) = \begin{cases} x+2 & \text{if } x \geq 0 \\ x-1 & \text{if } x \leq 4 \end{cases}$

Ans: Not a function; the cases overlap. For example, $G(1)$ is equal to both 3 and 0.

95. $f : \mathbf{R} \rightarrow \mathbf{R}$ where $f(x) = \begin{cases} x^2 & \text{if } x \leq 2 \\ x-1 & \text{if } x \geq 4 \end{cases}$

Ans: Not a function; $f(3)$ not defined.

96. $G : \mathbf{Q} \rightarrow \mathbf{Q}$ where $G(p/q) = q$.

Ans: Not a function; $f(1/2) = 2$ and $f(2/4) = 4$.

97. Give an example of a function $f : \mathbf{Z} \rightarrow \mathbf{Z}$ that is 1-1 and not onto \mathbf{Z} .

Ans: $f(n) = 2n$.

98. Give an example of a function $f : \mathbf{Z} \rightarrow \mathbf{Z}$ that is onto \mathbf{Z} but not 1-1.

Ans: $f(n) = \left\lfloor \frac{n}{2} \right\rfloor$.

99. Give an example of a function $f : \mathbf{Z} \rightarrow \mathbf{N}$ that is both 1-1 and onto \mathbf{N} .

Ans: $f(n) = \begin{cases} -2n & \text{if } n \leq 0 \\ 2n-1 & \text{if } n > 0 \end{cases}$